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A Study on Parametric Multiobjective Programming Problems Without Differentiability

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Abstract—Great efforts have been done for studying basic notions like the solvability and stability sets for parametric multiobjective programming problems of the first and second kind. In this paper, these notions have been defined and analyzed without differentiability assumptions on the considered functions. New stability notions like the stability sets of the third and fourth kind have been also defined and analyzed for this problem.

1. INTRODUCTION

Qualitative analysis of some basic notions like the set of feasible parameters, the solvability set and stability sets of the first and second kind were introduced by Osman in [1,2]. Also, in [3,4], the stability sets of the third and fourth kind for a general class of convex parametric programming problems were defined and analyzed. In addition, the relations between multiobjective programming problems and parametric programs have been studied in [5,6]. The stability of multiobjective programs using the Hybrid approach and using the extension of the generalized Tchebycheff norm were considered in [7,8]. In this paper, the stability notions of parametric multiobjective programming problems without differentiability assumptions are studied.

2. PROBLEM FORMULATION

Consider the general parametric vector optimization problem:

$$\mathrm{Vop}(\lambda)\begin{cases} \min F(\mathbf{x},\lambda) = (f_1(\mathbf{x},\lambda),\ f_2(\mathbf{x},\lambda),\dots,f_m(\mathbf{x},\lambda)), & \text{subject to} \\ M = \left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, g_r(\mathbf{x})\le 0, \quad r=1,2,\dots,k\right\}, \end{cases}$$

where $f_j(\mathbf{x},\lambda)$, $j = 1, 2, \dots, m$ and $g_r(\mathbf{x})$, $r = 1, 2, \dots, k$ are real valued functions convex on M and $\mathbb{R}^k$, respectively, and $\lambda \in \mathbb{R}^m$ is any vector parameter.

DEFINITION 1. *A point $\mathbf{x}^0 \in M$ is said to be an efficient solution of $Vop(\lambda)$ if there exists no other $\mathbf{x} \in M$ such that:*

$$f_j(\mathbf{x},\lambda) \le f_j(\mathbf{x}^0,\lambda) \quad \text{and} \quad f_j(\mathbf{x},\lambda) \ne f_j(\mathbf{x}^0,\lambda), \qquad j=1,2,\dots,m.$$

The scalarization of problem $\mathrm{Vop}(\lambda)$ can be written in the form:

$$\mathrm{Sop}(\lambda)\begin{cases} \min \sum_{j=1}^{m} w_j\, f_j(\mathbf{x},\lambda) & \text{subject to} \\ M = \left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, g_r(\mathbf{x})\le 0, \quad r=1,2,\dots,k\right\}, \end{cases}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX where

$$w_j \in \Lambda = \left\{ w \in \mathbb{R}^m \middle| w_j \geq 0, \quad j = 1, 2, \dots, m, \quad \sum_{j=1}^{m} w_j = 1 \right\}.$$

Here, we take $f_j(\mathbf{x}, \lambda) = \lambda_j f_j(\mathbf{x})$, $j = 1, 2, \dots, m$. It is well-known from [5], that the optimal solution $\mathbf{x}^*$ of Sop(λ) is an efficient solution of Vop(λ) if it has unique optimal solution.

Definition 2. *The solvability set of problem Vop(λ), which is denoted by B, is defined by*

$$B = \left\{ \lambda \in \mathbb{R}^m \middle| \text{problem } Vop(\lambda) \text{ has efficient solutions} \right\}.$$

Let

$$E(\lambda) = \left\{ \mathbf{x}^* \in \mathbb{R}^m \middle| \sum_{j=1}^{m} w_j \lambda_j f_j(\mathbf{x}^*) = \min_{\mathbf{x} \in M} \sum_{j=1}^{m} w_j \lambda_j f_j(\mathbf{x}) \right\}.$$

3. STABILITY SET OF THE FIRST KIND

Definition 3. *Suppose that for $\bar{\lambda} \in B$ an efficient solution of Vop(λ) is found to be $\bar{\mathbf{x}} \in E(\lambda)$; then the stability set of the first kind of Vop(λ) corresponding to $\bar{\mathbf{x}}$, which is denoted by $T(\bar{\mathbf{x}})$, is defined by*

$$T(\bar{\mathbf{x}}) = \left\{ \lambda \in B \middle| \bar{\mathbf{x}} \in E(\lambda) \text{ is an efficient solution of } Vop(\lambda) \right\}.$$

Theorem 1. *The set $T(\bar{\mathbf{x}})$ is convex.*

Proof. Let λ^1, $\lambda^2 \in T(\bar{\mathbf{x}})$, then

$$\sum_{j=1}^{m} w_j \lambda_j^1 f_j(\bar{\mathbf{x}}) \leq \sum_{j=1}^{m} w_j \lambda_j^1 f_j(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in M, \qquad \text{and}$$
$$\sum_{j=1}^{m} w_j \lambda_j^2 f_j(\bar{\mathbf{x}}) \leq \sum_{j=1}^{m} w_j \lambda_j^2 f_j(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in M.$$

Therefore,

$$\sum_{j=1}^{m} (\alpha w_j \lambda_j^1 + (1-\alpha) w_j \lambda_j^2) f_j(\bar{\mathbf{x}}) \leq \sum_{j=1}^{m} (\alpha w_j \lambda_j^1 + (1-\alpha) w_j \lambda_j^2) f_j(\mathbf{x}),$$
$$\text{for all } \mathbf{x} \in M \text{ and } 0 \leq \alpha \leq 1.$$

i.e., $\alpha \lambda_j^1 + (1-\alpha) \lambda_j^2 \in T(\bar{\mathbf{x}})$, $j = 1, 2, \dots, m$, $0 \leq \alpha \leq 1$, and hence the result follows. ∎

Theorem 2. *The set $T(\bar{\mathbf{x}})$ is a cone with vertex at $\lambda = 0$.*

Proof. It is clear that $\lambda = 0 \in T(\bar{\mathbf{x}})$. Suppose that $\tilde{\lambda} \in T(\bar{\mathbf{x}})$; then

$$\sum_{j=1}^{m} w_j \tilde{\lambda}_j f_j(\bar{\mathbf{x}}) \leq \sum_{j=1}^{m} w_j \tilde{\lambda}_j f_j(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in M, \text{ and then}$$
$$\sum_{j=1}^{m} w_j \alpha \tilde{\lambda}_j f_j(\bar{\mathbf{x}}) \leq \sum_{j=1}^{m} w_j \alpha \tilde{\lambda}_j f_j(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in M, \quad \alpha \geq 0,$$

i.e., $\alpha \tilde{\lambda} \in T(\bar{\mathbf{x}})$ for all $\alpha \geq 0$, and hence the result follows. ∎

4. STABILITY SET OF THE SECOND KIND

DEFINITION 4. *Suppose $\bar{\lambda} \in B$ and $I \subset \{1, 2, \ldots, m\}$. Let $\rho(\bar{\lambda}, I)$ denote the side of constraints defined by*

$$\rho(\bar{\lambda}, I) = \left\{ \mathbf{x} \in \mathbb{R}^n \middle| g_r(\mathbf{x}) = 0, \text{ for } r \in I; \text{ and } g_r(\mathbf{x}) < 0, \text{ for } r \notin I \right\},$$

then the stability set of the second kind of Vop(λ) corresponding to $\rho(\bar{\lambda}, I)$, denoted by $Q(\rho(\bar{\lambda}, I))$, is defined by

$$Q(\rho(\bar{\lambda}, I)) = \left\{ \lambda \in B \middle| Q(\rho(\bar{\lambda}, I)) \text{ contains an efficient solution of Vop}(\lambda) \right\}.$$

PROPOSITION 4.1. *If the functions $f_j(\mathbf{x})$, $j = 1, 2, \ldots, m$ are strictly convex on M, and $I_1 \neq I_2$, then*

$$Q(\rho(\lambda^1, I_1)) \cap Q(\rho(\lambda^2, I_2)) = \emptyset.$$

PROOF. Suppose that $\bar{\lambda} \in Q(\rho(\lambda^1, I_1)) \cap Q(\rho(\lambda^2, I_2))$, then

$$E(\bar{\lambda}) \cap \rho(\lambda^1, I_1) \neq \emptyset, \qquad \text{and}$$
$$E(\bar{\lambda}) \cap \rho(\lambda^2, I_2) \neq \emptyset,$$

which is a contradiction since $E(\bar{\lambda})$, by the assumption, is only a single point. ∎

REMARK 1. It follows from Definitions 3 and 4 that

$$Q(\rho(\lambda, I)) = \bigcup_{i \in J} S(\mathbf{x}^i),$$

where

$$J = \left\{ j \middle| \mathbf{x}^j \in (\bar{\rho}(\lambda, I)) \text{ is an efficient solution of Vop}(\lambda) \right\}.$$

∎

REMARK 2. If J is a finite set, then the set $Q(\rho(\bar{\lambda}, I)) \cup \{0\}$ is a closed cone (from Remark 1 and Theorem 2). ∎

THEOREM 3. *The set $Q(\rho(\bar{\lambda}, I)) \cup \{0\}$ is star shaped* [1], with the point $\lambda = 0$ as its common visibility point.

PROOF. If $\lambda \in Q(\rho(\bar{\lambda}, I))$, then from Remark 2, $\lambda \in T(\mathbf{x}^s)$ for at least one index $s \in J$. Then $\alpha\,\lambda \in T(\mathbf{x}^s) \cup \{0\}$, $\alpha \geq 0$ from the convexity of $T(\mathbf{x}^s)$, i.e., $\alpha\,\lambda \in Q(\rho(\bar{\lambda}, I))$, and hence the result follows. ∎

REMARK 3. From Definition 4, it follows that

$$Q(\bar{\rho}(\bar{\lambda}, I)) = \cup_{i \in \Gamma} Q(\rho(\lambda^i, I_i)),$$

where

$$\Gamma = \left\{ i \middle| I \leq I_i \quad \text{and} \quad \bar{\rho}(\bar{\lambda}, I) \text{ is the closure of } \rho(\bar{\lambda}, I) \right\}.$$

THEOREM 4. *If $f_j(\mathbf{x})$, $j = 1, 2, \ldots, m$ are continuous and strictly convex on M, and $Q(\rho(\bar{\lambda}, I)) \subset B \cup \{0\}$, then*

$$\bar{Q}(\rho(\bar{\lambda}, I)) \subset Q(\bar{\rho}(\bar{\lambda}, I)) \cup \{0\},$$

where $Q(\bar{\rho}(\bar{\lambda}, I))$ and $\bar{Q}(\rho(\bar{\lambda}, I))$ are, respectively, the boundary and the closure of $Q(\rho(\bar{\lambda}, I))$.

PROOF. If either $Q(\rho(\bar{\lambda}, I))$ is closed or $Q(\rho(\bar{\lambda}, I)) = S(\bar{\mathbf{x}})$, the result is clear.

Let λ^* be a boundary point of $Q(\rho(\bar{\lambda}, I))$; if $\lambda^* = 0$, the result is clear. Otherwise, choose a sequence $\lambda^{(n)} \geq 0$ which converge to λ^* such that $\lambda^{(n)} \in Q(\rho(\bar{\lambda}, I))$ with corresponding efficient solutions $\mathbf{x}^{(n)} \in Q(\rho(\bar{\lambda}, I))$. Then

$$\sum_{j=1}^{m} w_j\, \lambda_j^{(n)}\, f_j(\mathbf{x}^{(n)}) \leq \sum_{j=1}^{m} w_j\, \lambda_j^{(n)}\, f_j(\mathbf{x}),$$

for all $\mathbf{x} \in M$ and for all n. Therefore,

$$\lim_{n\to\infty} \sum_{j=1}^{m} w_j\, \lambda_j^{(n)}\, f_j(\mathbf{x}^{(n)}) \leq \lim_{n\to\infty} \sum_{j=1}^{m} w_j\, \lambda_j^{(n)}\, f_j(\mathbf{x}),$$

for all $\mathbf{x} \in M$. From the finiteness of the sum and the continuity of f_j, it follows that:

$$\sum_{j=1}^{m} w_j\, \lambda_j^*\, f_j(\lim_{n\to\infty} \mathbf{x}^{(n)}) \leq \sum_{j=1}^{m} w_j\, \lambda_j^*\, f_j(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in M, \qquad \text{i.e.,}$$

$$\sum_{j=1}^{m} w_j\, \lambda_j^*\, f_j(\mathbf{x}^*) \leq \sum_{j=1}^{m} w_j\, \lambda_j^*\, f_j(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in M,$$

where $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$ exists since $B \cup \{0\}$ is closed, and it is efficient by the fact that $E(\lambda^*) = \{\mathbf{x}^*\}$. Therefore, $\mathbf{x}^* \in \bar{\rho}(\bar{\lambda}, I)$ and this completes the proof. ∎

5. STABILITY SET OF THE THIRD KIND

DEFINITION 5. *Suppose that problem Vop(λ) is solvable with a corresponding efficient solution $\bar{\mathbf{x}}$, $\mathbf{x}^*$ is any feasible point, and $\delta > 0$; then the stability set of the third kind of Vop(λ), which is denoted by $K_3(\bar{\lambda}, \mathbf{x}^*, \delta)$, is defined by*

$$K_3(\bar{\lambda}, \mathbf{x}^*, \delta) = \left\{ \lambda \in \mathbb{R}^m \middle| \|F(\mathbf{x}^*, \lambda) - F(\bar{\mathbf{x}}, \bar{\lambda})\| < \delta \right\}.$$

LEMMA 1. *The set $K_3(\bar{\lambda}, \mathbf{x}^*, \delta)$ is convex.*

PROOF. Let $\lambda^1, \lambda^2 \in K_3(\bar{\lambda}, \mathbf{x}^*, \delta)$, then

$$\|F(\mathbf{x}^*, \lambda^1) - F(\bar{\mathbf{x}}, \bar{\lambda})\| < \delta, \quad \|F(\mathbf{x}^*, \lambda^2) - F(\bar{\mathbf{x}}, \bar{\lambda})\| < \delta.$$

Therefore,

$$\begin{aligned} (1-w)\, \|F((\mathbf{x}^*, \lambda^1) - F(\bar{\mathbf{x}}, \bar{\lambda})\| &< (1-w)\delta, \qquad \text{and} \\ w\, \|F((\mathbf{x}^*, \lambda^2) - F(\bar{\mathbf{x}}, \bar{\lambda})\| &< w\,\delta, \qquad 0 \leq w \leq 1. \end{aligned}$$

Hence,

$$\begin{aligned} &\|F((\mathbf{x}^*, (1-w)\, \lambda^1 + w\lambda^2) - F(\bar{\mathbf{x}}, \bar{\lambda})\| \\ &\qquad \leq (1-w)\, \|F(\mathbf{x}^*, \lambda^1) - F(\bar{\mathbf{x}}, \bar{\lambda})\| + w\, \|F(\mathbf{x}^*, \lambda^2) - F(\bar{\mathbf{x}}, \bar{\lambda})\| \\ &\qquad < (1-w)\, \delta + w\delta. \end{aligned}$$

Then $((1-w)\, \lambda^1 + w\, \lambda^2) \in K_3(\bar{\lambda}, \mathbf{x}^*, \delta)$, and hence the result follows. ∎

Now, under the assumption $F(\mathbf{x}, \lambda) = \lambda_j\, f_j(\mathbf{x})$, $j = 1, 2, \dots, m$, the determination of subset from the set $K_3(\bar{\lambda}, \mathbf{x}^*, \delta)$ is given as follows:

$$\begin{aligned} \|F(\mathbf{x}^*, \lambda) - F(\bar{\mathbf{x}}, \bar{\lambda})\| &= \|\lambda\, f(\mathbf{x}^*) - \bar{\lambda}\, f(\bar{\mathbf{x}})\| \\ &= \|\bar{\lambda}\, (f(\mathbf{x}^*) - f(\bar{\mathbf{x}})) + (\lambda - \bar{\lambda})\, f(\mathbf{x}^*)\| \\ &\leq \|\lambda\|\, \|f(\mathbf{x}^*) - f(\bar{\mathbf{x}})\| + \|\lambda - \bar{\lambda}\|\, \|f(\mathbf{x}^*)\| < \delta, \end{aligned}$$

i.e,

$$\|\lambda - \bar{\lambda}\| < \frac{\delta - \|\lambda\| \, \|f(\mathbf{x}^*) - f(\bar{\mathbf{x}})\|}{\|f(\mathbf{x}^*)\|} = \varepsilon.$$

If $I(\mathbf{x}^*)$ denotes the set

$$I(\mathbf{x}^*) = \left\{\lambda \in \mathbb{R}^m \middle| \|\lambda - \bar{\lambda}\| < \varepsilon\right\},$$

then $I(\mathbf{x}^*) \subset K_3(\bar{\lambda}, \mathbf{x}^*, \delta)$. In order that $I(\mathbf{x}^*) \neq \emptyset$, then it is clear that either δ is large or $\|f(\mathbf{x}^*) - f(\bar{\mathbf{x}})\|$ is sufficiently small.

REMARK 4. It must be noted that

$$I(\mathbf{x}^*) \subset T(\bar{\mathbf{x}}).$$

6. STABILITY SET OF THE FOURTH KIND

DEFINITION 6. *Suppose that the problem Vop(λ) is solvable at $\bar{\lambda} \in B$ with a corresponding efficient solution $\bar{\mathbf{x}}$, and $\delta > 0$, then the stability set of the fourth kind of Vop(λ), which is denoted by $K_4(\bar{\lambda}, \delta)$, is defined by*

$$K_4(\bar{\lambda}, \delta) = \left\{\lambda \in \mathbb{R}^m \middle| \text{ there exist } \mathbf{x} \in M, \quad \|F(\mathbf{x}, \lambda) - F(\bar{\mathbf{x}}, \bar{\lambda})\| < \delta\right\}.$$

LEMMA 2. *The set $K_4(\bar{\lambda}, \delta)$ is convex in λ and closed in $\mathbf{x}$.*

PROOF. The first part of the proof is clear from Lemma 1 at any feasible point $\mathbf{x} \in M$. To prove the second part, let $\tilde{x}_n \in K_4(\bar{\lambda}, \delta)$, $n = 1, 2, \ldots$ be a sequence of points which converges to $\tilde{x}$; then

$$\begin{aligned} &\|F(\tilde{x}_n, \lambda^*) - F(\bar{\mathbf{x}}, \bar{\lambda})\| < \delta, && \text{and} \\ &\|F(\tilde{x}_n, \lambda^*) - F(\tilde{x}, \lambda^*)\| \to 0 && \text{as } n \to \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} \|F(\tilde{x}, \lambda^*) - F(\bar{\mathbf{x}}, \bar{\lambda})\| &= \|F(\tilde{x}, \lambda^*) - F(\tilde{x}_n, \lambda^*) + F(\tilde{x}_n, \lambda^*) - F(\bar{\mathbf{x}}, \bar{\lambda})\| \\ &\le \|F(\tilde{x}, \lambda^*) - F(\tilde{x}_n, \lambda^*)\| + \|F(\tilde{x}_n, \lambda^*) - F(\bar{\mathbf{x}}, \bar{\lambda})\| < \delta, \end{aligned}$$

which means that $\tilde{x} \in K_4(\bar{\lambda}, \delta)$. Hence, the result follows. ∎

It is clear that $K_3(\bar{\lambda}, \mathbf{x}^*, \delta) \subset K_4(\bar{\lambda}, \delta)$. Following the same steps as those for determining $I(\mathbf{x}^*)$, it is clear that if we define

$$J = \cup_{\mathbf{x} \in M} I(\mathbf{x}),$$

then $J \subset K_4(\bar{\lambda}, \delta)$.

A simple expression for J can be deduced in the following case. If $f_j(\mathbf{x})$ are construction mappings on M, i.e., there exist a proper fraction p such that

$$\|f(\mathbf{x}^*) - f(\bar{\mathbf{x}})\| \le p \, \|\mathbf{x} - \bar{\mathbf{x}}\|,$$

then using Cauchy's inequality, it follows that

$$\|\lambda - \bar{\lambda}\| < \frac{\delta - p \, \|\bar{\lambda}\| \, \|\mathbf{x} - \bar{\mathbf{x}}\|}{\|f(\mathbf{x}^*)\|} = \gamma(\mathbf{x}).$$

If $I^{\backprime}(\mathbf{x})$ denotes the set

$$I^{\backprime}(\mathbf{x}) = \left\{\lambda \in \mathbb{R}^m \middle| \|\lambda - \bar{\lambda}\| < \gamma(\mathbf{x})\right\},$$

then

$$J^{\backprime} = \cup_{\mathbf{x} \in M} I^{\backprime}(\mathbf{x}).$$

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